Dense Inclined Flows of Inelastic Spheres: Tests of an Extension of Kinetic Theory

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Abstract: Using the results of recent numerical simulations, we extend an existing kinetic theory for dense flows of identical, nearly elastic, frictionless spheres to identical, very dissipative, frictional spheres. The existing theory incorporates an additional length scale in the expression for the collisional rate of dissipation; this length scale is identified with the size of a cluster of correlated particles. Parameters of the theory for very dissipative, frictional spheres are set using the results of physical experiments on inclined flows of spheres over a rigid, bumpy base in the absence of sidewalls. The resulting theory is then tested against the results of physical experiments on flows of the same material over the surface of an erodible heap when frictional sidewalls are present.

Keywords: Dense grain flow, Inclined flow, Inelastic spheres

Theory

Recent numerical simulations of dense granular shearing flows [1,2] indicate that correlations between the positions and/or the velocities of particles, not considered in simple kinetic theories for collisional flows, play an important role in determining the rheology. In a shear flow of dense, inelastic, compliant particles, the duration of a typical collision may equal or exceed the time between collisions. Then, simultaneous interactions between more than two particles become likely, both in discrete-element simulations that employ compliant spheres and in physical experiments. In this event, small groups of particles overlap [3-5] and/or interact through repeated, weak, "chattering" collisions [1]. Overlaps reduce the frequency of collisions while chattering replaces strong binary collisions with numerous weak ones. In a range of volume fractions between 0.49 and 0.60 and coefficients of restitution greater than 0.70, the influence of the correlations on the fluxes of momentum and energy is compensated for by non-local transport associated with the correlated motion. Consequently, overlaps and chattering first influence the collisional rate of dissipation. For denser and/or more inelastic flows, anisotropic and rate-independent contributions to the pressure and shear stress, associated with chains or clusters that eventually span the flow, are anticipated to develop [6-8].

Models of dense granular shearing flows have attempted to incorporate these correlations in various ways. Ertas and Halsey [9] introduce a length associated with vorticial structures, Kumaran [10] employs higher spatial gradients in a kinetic theory for collisional interactions in dense shearing flows, Mitarai and Nakanishi [1] focus on a modification of the strength of the particle agitation that is associated with correlated velocities, and Jenkins [11,12] introduces a length associated with the size of particle clusters into the expression for the rate of collisional dissipation.

Here, we attempt to improve the predictive capability of Jenkins' [12] theory and extend it to very dissipative, frictional spheres. We first incorporate the observed frictional behavior in an effective coefficient of restitution. Then, because this results in description that involves substantial dissipation, we adopt the kinetic theory of Garzo and Dufty [13] for frictionless but very inelastic spheres. When applying the theory to dense flows, we treat overlapping and chattering particles in the same way; assume, as indicated by the numerical simulations [1,2], that the singularity in the radial distribution function occurs at volume fractions less than that for random dense packing; and choose the two parameters in Jenkins' [12] theory to provide a reasonable agreement with the results of Pouliquen's [14] physical experiments on inclined flows. We then use these material parameters and a simple, algebraic form of the energy balance to predict the relations between volume flux, flow depth, and angle of inclination in dense granular flows between friction sidewalls over the inclined surface of a heap and compare the predictions with the measurements of Jop, et al. [15].

Balance equations

We consider identical spheres with a mass m and a diameter d. The mean number of particles per unit volume is n, the mean velocity **u** is defined as the average over particle velocity **c** using the single particle velocity distribution function,

$\mathbf{u} \equiv \langle \mathbf{c} \rangle$, the fluctuation velocity is then $\mathbf{C} \equiv \mathbf{c} - \mathbf{u}$, and the strength of the velocity fluctuations, the granular temperature, is defined in terms of **C** by $T \equiv \langle C^2 \rangle / 3$.

The mass density $\rho \equiv mn$, the mean velocity, and the granular temperature are determined as solutions to the balance of mass and linear momentum that have their usual forms (e.g., [16]), and the balance of fluctuation energy,

$$(3/2)\rho\dot{T} = \mathrm{tr}\left(\mathbf{tD}\right) - \nabla \cdot \mathbf{q} - \Gamma . \tag{1}$$

That is, the convected time rate of change of internal energy following the mean motion is equal to the rate at which fluctuations are created by the working of the stress **t** through the symmetric part **D** of the gradients of the mean velocity, less the sum of the divergence of the flux of fluctuation energy **q** and the rate of collisional dissipation Γ.

Numerical simulations of homogeneous, steady shearing flows of granular materials [17,18] indicate that the presence of friction does have an influence on the tractions necessary to maintain a shearing flow at a given solids fraction. Here, rather that introduce additional balance laws for angular momentum and rotational fluctuation energy, we attempt to take into account the frictional interactions only through their influence on the energy of the fluctuations of the translation velocity [19,20].

Constitutive Relations

For solid volume fractions $\nu \equiv n\pi d^3/6$ less than 0.49, we employ the constitutive relations for shearing flows that result from the kinetic theory of Garzo and Dufty [13] for identical frictionless, dissipative elastic spheres, but do not incorporate the small terms introduced by their function $c*$ of the coefficient of restitution. The magnitude of c* is less than 0.4 and terms proportional to c* are typically multiplied by a small numerical coefficient.

If the x coordinate is taken in the flow direction, the y coordinate taken in the direction of shear, and the z coordinate orthogonal to these, the pressure $p \equiv -\left(t_{xx} + t_{yy} + t_{zz}\right)/3$, the shear stress $S \equiv t_{xy}$, the energy flux $Q \equiv q_y$, and the rate of collisional dissipation Γ are given by

$$p = 4\rho GFT , \tag{2}$$

where $G \equiv \nu(1-\nu/2)/(1-\nu)^3$ is the product of ν and the expression for the volume fraction dependence of the radial distribution function at contact determined by Carnahan and Starling [21] in numerical simulations at moderate volume fractions and $F = (1+e)/2 + 1/(4G)$;

$$S = \mu u', \tag{3}$$

where the prime denotes a derivative with respect to y and

$$\mu = \left(2J/5\pi^{1/2}\right) pd / \left(FT^{1/2}\right), \tag{4}$$

with

$$J = \frac{(1+e)}{2} + \frac{\pi}{32} \frac{\left[5+2(1+e)(3e-1)G\right]\left[5+4(1+e)G\right]}{\left[24-6(1-e)^2-5(1-e^2)\right]G^2}; \tag{5}$$

$$Q = -\kappa T' - \eta \nu', \tag{6}$$

where

$$\kappa = \left(M/\pi^{1/2}\right) pd / \left(FT^{1/2}\right), \tag{7}$$

with

$$M = \frac{1+e}{2} + \frac{9\pi}{144} \frac{\left[5+3G\left(1+e\right)^2\left(2e-1\right)\right]\left[5+6G\left(1+e\right)\right]}{\left(1+e\right)\left[16-7\left(1-e\right)\right]G^2} \tag{8}$$

and

$$\eta = \frac{25\pi^{1/2}}{128} \frac{pT^{1/2}}{4FG} \frac{d}{\nu^2} N \tag{9}$$

with

$$N = \frac{96}{25} \frac{(1-e)}{(1+e)} \frac{\nu}{G} \left[5+6G(1+e)\right]$$

$$\times \frac{20\nu(\ln G)_\nu \left[5+3(1+e)^2(2e-1)\right] / \left[48-21(1-e)\right] - G\left[1+\nu(\ln G)_\nu (1+e)e\right]}{16+3(1-e)}$$

$$\tag{10}$$

in which the subscript indicates a derivative with respect to ν; and

$$\Gamma = \frac{12}{\pi^{1/2}} \frac{\rho G}{d} (1 - e^2) T^{3/2} . \tag{11}$$

To incorporate the additional dissipation due to friction, we make use of a calculation by Herbst, et al. [22] of the rates of dissipation of the rotational and translational energy in a steady, homogeneous shearing flow. Because the numerical simulations indicate that the tractions quickly reach limiting values as the coefficient of sliding friction increases above 0.10, we apply their calculation to spheres in the limit of infinite friction. In this case, the ratio R of the energy of the rotational an translation velocity fluctuations is given by

$$R = \frac{2(1+\beta_0)}{14 - 5(1+\beta_0)} , \tag{12}$$

where β_0 is the coefficient of tangential restitution in a sticking collision (e.g., [23]), and the rate of dissipation of translation fluctuation energy is

$$\Gamma = \frac{48}{\pi^{1/2}} \frac{\rho G}{d} \left[\frac{1-e^2}{4} + \frac{1+\beta_0}{7} - \left(\frac{1+\beta_0}{7} \right)^2 \left(1 + \frac{5}{2} R \right) \right] T^{3/2} . \tag{13}$$

Consequently,

$$\frac{1-e_{\mathrm{eff}}^2}{4} \equiv \frac{1-e^2}{4} + \frac{1+\beta_0}{7} - \left(\frac{1+\beta_0}{7} \right)^2 \left[1 + \frac{5(1+\beta_0)}{14 - 5(1+\beta_0)} \right] . \tag{14}$$

From this point onward, we use the symbol e to denote the effective coefficient of restitution.

When the volume fraction is greater than 0.49, but less than 0.60, the constitutive relations are taken to be those given above in the limit that the terms proportional to 1/G are small compared to unity, the volume fraction dependence in G is taken to be a modification of that determined by Torquato [24] in simulations of dense aggregates of spheres, and the particle diameter d in the rate of collisional dissipation is replaced by the length L of a typical chain of contacting particles. Then

$$p = 2(1+e)\rho G T , \tag{15}$$

where $G = 0.63\nu / \left(0.60 - \nu\right)$; the expressions for S and μ are unchanged, but $J = (1+e)/2 + (\pi/4)(3e-1)(1+e)^2 / \left[24 - (1-e)(11-e)\right]$; the expressions for Q and κ are unchanged, but $M = (1+e)/2 + (9\pi/8)(2e-1)(1+e)^2 / \left[16 - 7(1-e)\right]$; and

$$\Gamma = \frac{12}{\pi^{1/2}} \frac{\rho G}{L} (1 - e^2) T^{3/2} . \tag{16}$$

We expect that the volume fraction at which G becomes singular depends upon the coefficient of restitution; the value of 0.60 seems to be appropriate for a coefficient of restitution of 0.70 (e.g., [1]). At volume fractions above 0.60, we anticipate that shear rigidity develops and that this contributes a rate-independent term to the shear stress and pressure.

Chain Length

We assume that the spheres are forced into overlaping or chattering contact along the principal compressive axis of the shearing flow and that the random motion of the spheres acts to destroy this order. The principal compressive axis is the eigenvector of the strain rate **D** that is associated with its most negative eigenvalue. Then, the magnitude and direction of the vector **L** of chain length is determined by the simple balance

$$\hat{c} G^{1/3} D_{ik} L_k + (L T^{1/2} / d^2) L_i = 0 , \tag{17}$$

where $\hat{c}$ is a constant of order one. The power of G has been chosen to be 1/3 rather than 1/2 [12], because, as will be seen, this power, together with the value $\hat{c} = 0.50$, provides a relatively good fit to the physical experiments of Pouliquen [14].

The relation (17) is a rough, microscopic balance between the effect that is conjectured to create chains or clusters - particles being forced together by the mean shearing into correlated interactions along the principal compressive axis that persist for a time equal to the inverse of the shear rate - and the effect that destroys them - collisions between particles in directions other than that of the principal axis of compression. The balance is phenomenological and crude by the standards of the kinetic theory that it is used with. However, when employed in conjunction with the algebraic form of the energy balance and the constitutive relations of the kinetic theory, the resulting theory has been shown [11,12] to reproduce the qualitative features of inclined flows seen in numerical simulations. Here, we show that with appropriate choices for the constant $\hat{c}$, power of G, and coefficient of restitution e, it reproduces the quantitative relations between volume flow rates, flow depths, and angles of inclination measured in physical experiments.

In a planar shearing flow, (17) yields

$$\frac{L}{d}=\frac{1}{2}\hat{c}G^{1/3}\frac{du'}{T^{1/2}}\,. \tag{18}$$

With S given by (3) and (4) and $F=(1+e)/2$ in the dense limit;

$$\frac{du'}{T^{1/2}}=\frac{5\pi^{1/2}}{2J}\frac{(1+e)}{2}\frac{S}{p}\,; \tag{19}$$

so L/d may be expressed in terms of the stress ratio and the volume fraction as

$$\frac{L}{d}=\frac{5\pi^{1/2}}{4J}\frac{(1+e)}{2}\frac{S}{p}\hat{c}G^{1/3}\,. \tag{20}$$

Upon employing the algebraic approximation to the energy balance

$$Su'-\Gamma=0 \tag{21}$$

with Γ given by (16), we find that

$$\frac{\left(du'\right)^2}{T}=\frac{15}{2J}\frac{d}{L}\left(1-e^2\right). \tag{22}$$

Using (18) to eliminate L gives,

$$\left(\frac{du'}{T^{1/2}}\right)^3=\frac{15}{J}\frac{(1-e^2)}{\hat{c}G^{1/3}}\,. \tag{23}$$

With this, L/d and S/p can be expressed as functions of ν and e as

$$\frac{L}{d}=\frac{1}{2}\left[\frac{15}{J}(1-e^2)\hat{c}^2\right]^{1/3}G^{2/9} \tag{24}$$

and

$$\frac{S}{p}=\frac{4J}{5\pi^{1/2}}\frac{1}{(1+e)}\left[\frac{15}{J}\frac{(1-e^2)}{\hat{c}}\right]^{1/3}\frac{1}{G^{1/9}}\,. \tag{25}$$

Inverting (25) and using the relation between G and ν,

$$\nu=0.60\left\{1+0.63\left[\frac{15}{\pi^{3/2}}\left(\frac{4}{5}\right)^3\frac{J^2}{\hat{c}}\frac{\left(1-e\right)}{\left(1+e\right)^2\mu^3}\right]^{-3}\right\}^{-1} \tag{26}$$

This description can be phrased as a relationship between the stress ratio and the inertial parameter $I \equiv du'/(p/\rho)^{1/2}$ introduced by GRD MiDi [17]. Upon combining (15) and (19), we obtain

$$\frac{du'}{(p/\rho)^{1/2}} = \frac{5}{4J}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{1+e}{G}\right)^{1/2}\frac{S}{p}. \qquad (27)$$

With (25), this yields the relation

$$\frac{S}{p} = \left\{\frac{5}{4}\left(\frac{\pi}{2}\right)^{1/2}\left[\frac{25\pi^{3/2}}{192}\frac{\hat{c}}{J^{8/3}}\frac{(1+e)^{7/3}}{(1-e)}\right]^{3/2}\right\}^{-2/11} I^{2/11}. \qquad (28)$$

Equations (26) and (28) specify the volume fraction and the stress ratio as functions of the inertial parameter, the coefficient of restitution, and the parameter c. These relations are equivalent to those proposed by GDR Midi [17] over the range of volume fractions and coefficients of restitution for which exchanges of momentum in collisions dominate the momentum transfer and before force chains span the system. That is, as ν increases for a given e, the chain length L given by (24) will approach the system size. When it does, there is an additional mechanism for the transfer of momentum in the flow that we do not consider (e.g., [8]) - ephemeral chains of particles that transfer force across the flow and are responsible for the development of a yield stress. The model of GDR Midi continues to apply above this volume fraction and includes the rate-independent mechanism of momentum transfer, but the model developed here does not.

Boundary-Value Problem

We next use the balance laws and constitutive relations outlined above to phrase and solve a boundary-value problem for the steady, fully-developed flow over a rigid, bumpy base inclined at an angle θ to the horizontal and compare the results with quantities measured in physical experiments by Pouliquen [14].

Differential equations

The balances of momentum parallel and perpendicular to the flow are

$$S' = -\rho g \sin\theta, \qquad (29)$$

where

$$u' = S/\mu, \tag{30}$$

with μ given by (4) and (5); and

$$p' = -\rho g \cos\theta, \tag{31}$$

where, upon carrying out the differentiation and using (6), this first order equation for p can be employed as a first order equation for ν:

$$\left\{T\left[\rho(1+4G)\right]_{\nu} - \eta/\kappa\right\}\nu' = \rho(1+4G)Q/\kappa - \rho g \cos\theta, \tag{32}$$

in which η is given by (9) and (10) and κ is given by (7) and (8).

The energy balance is

$$Q' = Su' - \Gamma, \tag{33}$$

where Γ is given by (16), with $L = d$, if $15(1-e^2)\hat{c}^2 G^{2/3}/(8J) \leq 1$ and $L = d(5\pi^{1/2}/8J)(1+e)\hat{c}G^{1/3}S/p$, otherwise; and

$$T' = -\left(Q + \eta\nu'\right)/\kappa, \tag{34}$$

where ν' is given in (32).

The specification of the volume V of the material over a unit area of the base is implemented as a boundary condition to a first order differential equation for the partial hold-up, $v(y) \equiv \int_0^y \nu(\xi)d\xi$:

$$v' = \nu, \tag{35}$$

with $v(0) = 0$ and $v(H) = V$.

Boundary conditions

We assume that the base consists of a flat wall to which spheres identical to those in the flow have been fixed and use the boundary conditions on the slip velocity and energy flux at a bumpy, nearly elastic, frictionless base derived by Richman [25] with the effective coefficient of restitution. More complicated expressions that capture more precisely the influence of the friction [26] on the slip and energy flux are available.

The balance of momentum tangent to the base provides

$$\frac{u}{T^{1/2}} = \left(\frac{\pi}{2}\right)^{1/2} f \ \frac{S}{p}, \tag{36}$$

where

$$f = \frac{3}{2^{5/2} J} \frac{2^{3/2} J - 5F(1+B)\sin^2 \psi}{\left[2(1-\cos\psi)/\sin^2\psi - \cos\psi\right]} + \frac{5F}{2^{1/2} J}, \tag{37}$$

where $B = \left[1 + 5/(8G)\right]\pi/(2^{1/2}12)$, and ψ, the bumpiness, measures the average maximum penetration of a flow sphere between boundary spheres. When the diameter of the flow spheres is the same as that of the boundary spheres, the bumpiness is given in terms of d and the average separation s between the edges of the boundary spheres by $\sin\psi = (d+s)/(2d)$.

The balance of energy at the base is

$$Su = Q + D, \tag{38}$$

where $D = \left(2/\pi\right)^{1/2} pT^{1/2} h(1-e)$, with $h = 2(1-\cos\psi)/\sin^2\psi$.

We take the top of the flow to be the point where the free flight trajectory of a particle ejected normal to the flow with velocity $T^{1/2}$ first equals the mean free distance between collisions. At this point [27],

$$p = 4\rho GFT = 0.037\rho gd/\nu. \tag{39}$$

The flow momentum and energy flux there, associated with the acceleration of the particle under gravity [28], are

$$S = p\tan\theta \tag{40}$$

and

$$Q = -pT^{1/2}\tan^2\theta. \tag{41}$$

We take $e = 0.60$, $\psi = \pi/3$, and $c = 0.50$ and employ the two-point Matlab boundary-value problem solver bvp4c to determine solutions for ranges of dimensionless hold-up at three values of inclination in Pouliquen's [14] experiments. As far as we know, the continuum equations that result from the kinetic theory of Garzo & Dufty have not previously been used to solve a boundary-value problem. The introduction of the additional dissipation as an effective coefficient of translational restitution in the limit of high friction coefficient is also new; effective coefficients of restitution have previously been calculated in the limit of small friction coefficient [19,20]. Torquato's radial distribution functions with the singularities at random close packing have previously been employed for dense shearing flows of nearly elastic, frictionless disks [11] and spheres [12].

In Fig. 1, we show the predicted dimensionless depth-averaged velocity versus the dimensionless depth of flow versus the measurements of Pouliquen [14]. The agreement is rather good. In Fig. 2, we show profiles of volume fraction for various depths at the three angles of inclination. The profiles of volume fraction at the lowest angle of inclination exhibit the features seen in numerical simulations [29,30]: there are solutions over a range of volume hold-ups and the individual profiles of volume fraction exhibit a core in which the volume fraction is constant. At the higher angles of inclination, the base influences the flow throughout its depth; but, at a given angle of inclination, flows are still possible over a range of depths. The value of the effective coefficient of restitution that we employ for the glass spheres used by Pouliquen [14] corresponds to values of $e = 0.92$ and $\beta_0 = 0.25$ for the true restitution coefficients that are close to those measured by Foerster, et al. [23].

Algebraic Theory

The numerical simulations of Mitarai & Nakanishi [30] show that the profile of the granular temperature T in a dense inclined flow may be determined using the algebraic balance between production and dissipation of fluctuation energy and Jenkins [11,12] confirmed this in the context of his extension of the kinetic theory.
Here, we employ the constitutive theory outlined above and the algebraic balance of the energy to study flows over the surface of a heap that is confined between parallel, vertical walls and compare the predictions with the results of physical experiments by Jop, et al. [15].

Inclined, erodible flow with sidewalls

We take sliding friction of the side walls, with friction coefficient μ_w, into account in an approximate way by including the average frictional resistance of the sidewalls through the thickness W of the flow in a one-dimensional analysis through the depth of the flow. At the base of the flow, we assume that the material yields; this fixes the ratio of shear stress to pressure there. We assume that the flow is so dense that the extension of the kinetic theory that involves the additional length scales applies throughout its depth. We will see that this limits the angles of inclination for which the analysis applies. For greater angles of inclination, there is a more dilute region above the dense region that is described by the classical kinetic theory. Here we assume that the upper surface is free and show only these results. They include essentially all of the situations considered by Jop, et al. [15].

In a steady, fully-developed, inclined flow in a dense layer of thickness H, because the mass density is nearly constant, the pressure p is given to a good approximation by

$$p \doteq \rho g (H - y) \cos\phi , \tag{42}$$

where $\rho \doteq 0.60\rho_s$, with ρ_s the mass density material of the spheres, g is the gravitational acceleration, y is measured normal to the base and upwards from it, and θ is the angle of inclination. With $p = 2\rho(1+e)GT$ from (15),

$$T = \frac{1}{2(1+e)} \frac{p}{\rho G} = \frac{1}{2(1+e)} \frac{g(H-y)\cos\theta}{G} , \tag{43}$$

where, from (25),

$$G = \left[\frac{192}{25\pi^{3/2}} \frac{J^2}{\hat{c}} \frac{(1-e)}{(1+e)^2} \left(\frac{p}{S} \right)^3 \right]^3 . \tag{44}$$

In the flow,

$$\frac{dS}{dy} = -\rho g \sin\theta + 2\mu_w \frac{p}{W} \tag{45}$$

and

$$\frac{dp}{dy} = -\rho g \cos\theta ; \tag{46}$$

so,

$$\frac{dS}{dp} = \tan\theta - 2\mu_w \frac{p}{\rho g W \cos\theta} , \tag{47}$$

or

$$S = p\tan\theta - \mu_w \frac{p^2}{\rho g W \cos\theta} + \text{Const.} \tag{48}$$

At the base of the flow the material is assumed to be at yield. Then, if the shear stress and pressure there are denoted by the subscript zero: $S_0 = \alpha p_0$, where α is known, and $p_0 = \rho g H \cos\theta$. With this, (48) becomes

$$\frac{S}{p} = \alpha \frac{p_0}{p} - \left(\frac{p_0}{p} - 1\right)\tan\theta + \mu_w \left[\left(\frac{p_0}{p}\right)^2 - 1\right]\frac{p}{\rho g W \cos\theta}. \qquad (49)$$

When the upper surface is free, the height of the flow follows from (49):

$$0 = \alpha - \tan\theta + \mu_w \frac{H}{W} \qquad (50)$$

The velocity u satisfies

$$\frac{du}{dy} = \frac{5\pi^{1/2}}{4J}(1+e)\frac{T^{1/2}}{d}\frac{S}{p}. \qquad (51)$$

With (46), this may be written as

$$-\rho g \cos\theta \frac{du}{dp} = \frac{5\pi^{1/2}}{4J}(1+e)\frac{T^{1/2}}{d}\frac{S}{p}, \qquad (52)$$

where the dependence of $T^{1/2}$ on p is explicit in (43) and also enters through the dependence in (44) of G on S/p and the dependence in (49) of S/p on p. As an approximation, the dependence of G is ignored and, in (52), the value $\bar{G}$ of G evaluated at the average value $\overline{S/p}$ of S/p is employed, where

$$\frac{\bar{S}}{p} = \tan\theta - \frac{\mu_w}{2}\frac{H}{W} . \qquad (53)$$

Then, (52) may be integrated with the boundary condition $u(p_0) = 0$ to give

$$\frac{u}{(gd\cos\phi)^{1/2}} = -\left[\frac{\pi}{72}\frac{(1+e)}{J^2\bar{G}}\right]^{1/2}\left(\frac{p_0}{\rho g d\cos\theta}\right)^{3/2}\left\{15\alpha\left[\left(\frac{p}{p_0}\right)^{1/2} - 1\right]\right.$$

$$\left. + 5\left[2 - 3\left(\frac{p}{p_0}\right)^{1/2} + \left(\frac{p}{p_0}\right)^{3/2}\right]\tan\theta - 3\mu_w\left[4 - 5\left(\frac{p}{p_0}\right)^{1/2} + \left(\frac{p}{p_0}\right)^{5/2}\right]\frac{H}{W}\right\}$$

(54)

The depth-averaged velocity U based on this is given by

$$\frac{U}{(gd\cos\phi)^{1/2}} = \left[\frac{\pi}{72}\frac{(1+e)}{J^2\bar{G}}\right]^{1/2}\left(\frac{H}{d}\right)^{3/2}\left(5\alpha - 2\tan\theta - \frac{20}{7}\mu_w\frac{H}{W}\right). \qquad (55)$$

If we assume that the volume fraction throughout the flow is 0.60, the volume flow rate per unit width, q, is

$$\frac{q}{d(gd\cos\theta)^{1/2}} = 0.60\frac{H}{d}\frac{U}{(gd\cos\theta)^{1/2}} . \tag{56}$$

Finally, if the top of the layer is defined as the point at which L=1, then upon eliminating G between (20) and (25) and setting L=1,

$$\left(\frac{S}{p}\right)^2 = \frac{24}{5\pi}\frac{(1-e)}{(1+e)} . \tag{57}$$

This limits the inclination of the flow that can be considered; for $e = 0.60$, $S/p = \tan\theta$ cannot be greater than 0.58

In Fig. 3, we compare the predictions of (56) and (57) with the values measured by Jop, et al. [14] using a coefficient of sidewall friction of 0.25 and a yield stress ratio of 0.40. The agreement is good, despite the number of approximations that have been made in the analysis. However, the height of the flow at the various angles of inclination is under-predicted by as much as forty per cent.

Conclusions

Kinetic theory for very dissipative, frictional spheres, extended to include an additional length scale associated with the cluster size, can predict the experimental results of Pouliquen [14] on dense, inclined flows over a rigid bed without side walls. The associated algebraic theory used with the same parameters can reproduce at least some of the experimental results of Jop, et al. [15] on dense, inclined flows over an erodible bed between frictional sidewalls.

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Figure Captions

Fig. 1: Predicted (lines) and measured (symbols) dimensionless depth-averaged velocity versus dimensionless height for angles of inclination of 28^{o} (full line, circles) 25^{o} (dashed line, squares), and 22^{o} (dashed-dot line, triangles) for $e = 0.60$, $\hat{c} = 0.50$, and $\psi = \pi / 3$.

Fig. 2: Volume fraction versus dimensionless depth for flows of various heights at angles of inclination of 28^{o} (full lines) 25^{o} (dashed lines), and 22^{o} (dashed-dot lines) for $e = 0.60$, $\hat{c} = 0.50$, and $\psi = \pi / 3$.

Fig. 3: Predicted (lines) and measured (symbols) values of the dimensionless volume flow rate versus the tangent of the angle of inclination for three flow widths.

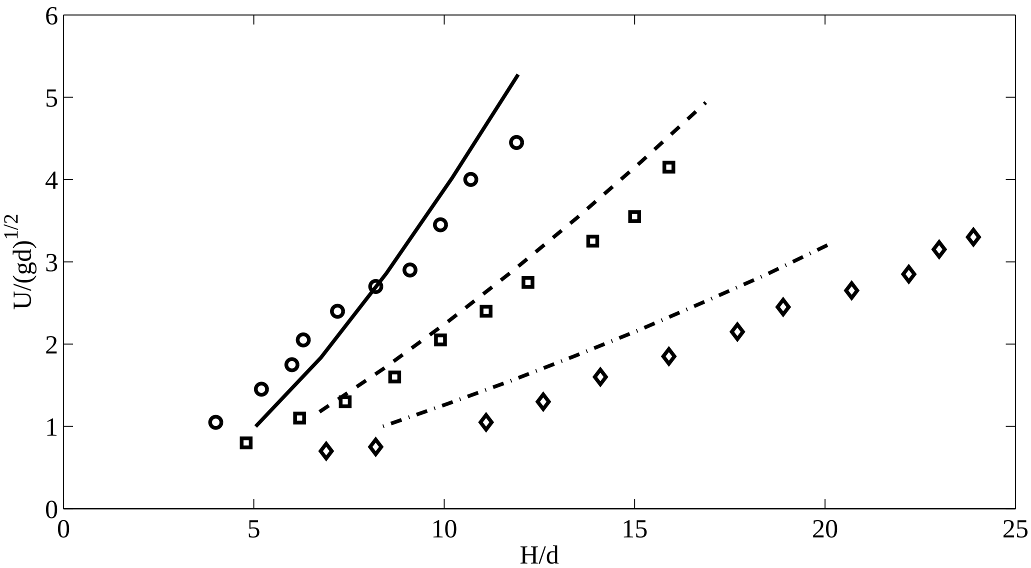

Figure 1

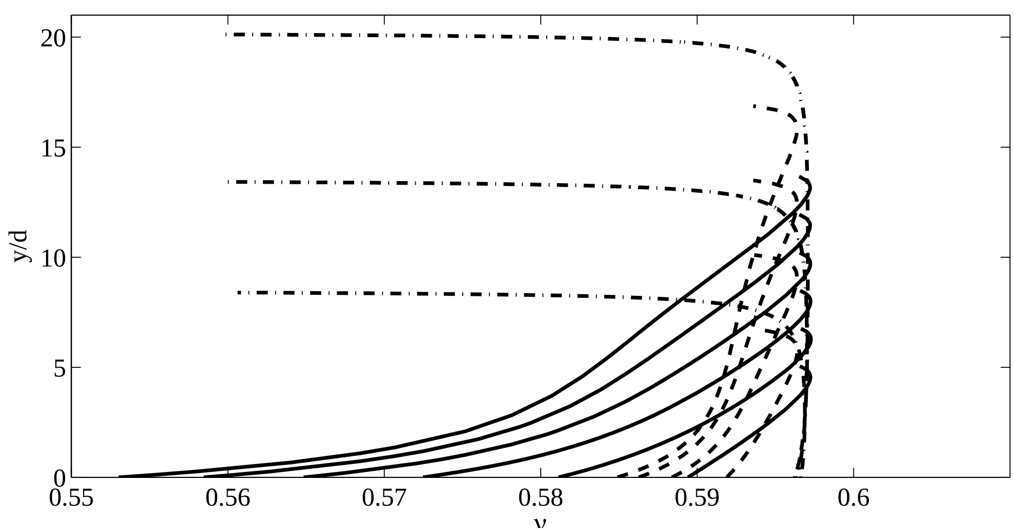

Figure 2

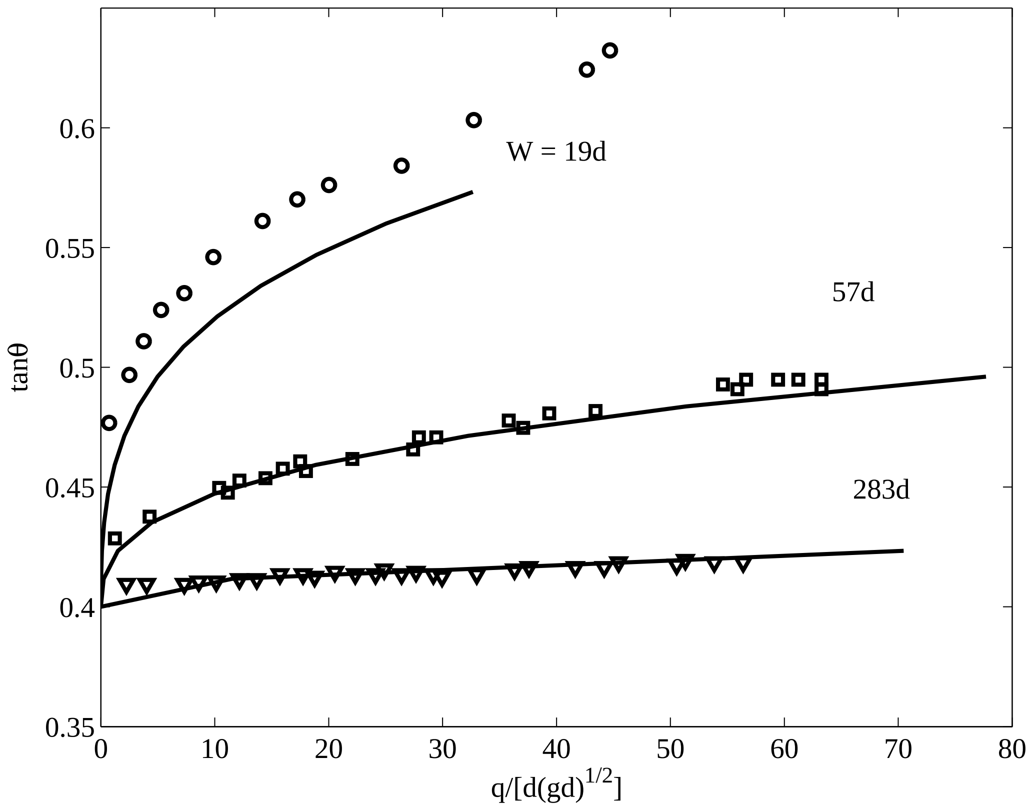
W = 19d
57d
283d
tanθ
q/[d(gd)^1/2]
0.35
0.4
0.45
0.5
0.55
0.6
0
10
20
30
40
50
60
70
80

Figure 3